Renormalization Groups on Fermions (I) [Shanker]

- 3 "theories" of Fermi liquid: Landau, $\underbrace{\text{Landau-er}}_{\text{Landau's students}}$, $\underbrace{\text{Landau-est}}_{\text{modern}}$

- Renormalization is possible whenever the physics is described by a partition function Z.

- Example: Bosonic oscillator

$$H = \omega_0 a^\dagger a \quad ; \quad \mathcal{U} = e^{-iHt/\hbar}, \quad Z = \mathrm{Tr}\, e^{-\beta H}$$

Now, $e^{-\beta H} = (e^{-\frac{\beta}{N}H})^N \approx (1-\epsilon H)^N$ $\quad \left[\epsilon = \frac{\beta}{N}\right]$

Introduce partition of identity: $\begin{cases} I \propto \int dz d\bar{z}\, |z\rangle\langle \bar{z}| e^{-\bar{z}z}, & a|z\rangle = z|z\rangle \\ \langle \bar{z}'|z\rangle = e^{\bar{z}'z}, & \langle z|a^\dagger = \langle z|\bar{z} \end{cases}$

$$\Rightarrow Z = \prod d\bar{z}(\tau)\, dz(\tau) \exp\left[\sum\left(\left(\frac{\bar{z}(\tau+\epsilon)-\bar{z}(\tau)}{\epsilon}\right) z\epsilon - \epsilon \bar{z} z \omega_0\right)\right]$$
$$= \int \mathcal{D}\bar{z}\, \mathcal{D}z\, e^{\int_0^\infty \bar{z}\left(\frac{\partial}{\partial \tau} - \omega_0\right) z\, d\tau}$$
$$= \int \mathcal{D}\bar{z}(\omega)\, \mathcal{D}z(\omega)\, e^{\int \bar{z}(\omega)(i\omega - \omega_0) z(\omega) d\omega}$$

$$\Rightarrow \langle \bar{z}(\omega_1)\, z(\omega_2)\rangle = \frac{\int \mathcal{D}\bar{z}\, \mathcal{D}z\, \bar{z} z\, e^{-(\dots)}}{Z} = \frac{\delta(\omega-\omega')}{i\omega - \omega_0}$$
$$\langle \bar{z}(4)\, \bar{z}(3)\, z(2)\, z(1)\rangle = \langle \bar{4}\, 1\rangle\langle \bar{3}\, 2\rangle + \langle \bar{4}\, 2\rangle\langle \bar{3}\, 1\rangle$$

- Example: Fermionic oscillator

$$H = \psi^\dagger \psi\, \omega_0 \quad ; \quad \{\psi, \psi^\dagger\} = 1, \quad \{\psi,\psi\} = \{\psi^\dagger, \psi^\dagger\} = 0.$$

Again we want resolution of identity, but we need:

$\psi|\chi\rangle = \chi|\chi\rangle$, $\quad \psi^2|\chi\rangle = \chi^2|\chi\rangle = 0 \quad \Longrightarrow \chi$ Grassmann variable.

$$\Rightarrow Z = \int \mathcal{D}\bar{\psi}(\omega)\, \mathcal{D}\psi(\omega)\, e^{\int \bar{\psi}(\omega)(i\omega-\omega_0)\psi(\omega) d\omega}$$
$$\Rightarrow \langle \bar{\psi}(\omega_1)\, \psi(\omega_2)\rangle = \frac{\delta(\omega-\omega')}{i\omega - \omega_0}$$
$$\langle \bar{4}\, \bar{3}\, 2\, 1\rangle = \langle \bar{4}\, 1\rangle\langle \bar{3}\, 2\rangle - \langle \bar{4}\, 2\rangle\langle \bar{3}\, 1\rangle \qquad [\text{Wick's thm}]$$

- More complicated example: $H = \sum_i \psi_i^\dagger \psi_i \omega_i + (\psi_i^\dagger \psi_i)(\psi_j^\dagger \psi_j)$

$$\Longrightarrow Z = \int \mathcal{D}\bar{\psi}(\omega)\, \mathcal{D}\psi(\omega) \exp\left[\sum_i \int \left(\bar{\psi}_i (i\omega - \omega_0)\psi_i - \bar{\psi}_i \bar{\psi}_j \psi_i \psi_j\right) d\omega\right)$$